Nonparametric Methods: Analysis of Ordinal Data

Chapter 16

Learning Objectives

LO16-1 Use the sign test to compare two dependent populations

LO16-2 Test a hypothesis about a median using the sign test

LO16-3 Test a hypothesis of dependent populations using the Wilcoxon signed-rank test

LO16-4 Test a hypothesis of independent populations using the Wilcoxon rank-sum test

LO16-5 Test a hypothesis of several independent populations using the Kruskal-Wallis test

LO16-6 Test and interpret a nonparametric hypothesis test of correlation

The Sign Test

- The sign test is based on the sign difference between two related observations
- No assumptions need to be made about the shape of the two populations
- For small samples, find the number of + or – signs and refer to the binomial distribution for the critical value
- Example
- A dietitian wishes to see if by taking a certain mineral, a person's cholesterol level decreases
- She measures the individuals before and after
- If there has been an decrease "+", an increase "-"

The Sign Test Example

The director of information systems at Samuelson Chemicals recommended that an in-plant training program be instituted for certain managers in Payroll, Accounting, and Production Planning. A sample of 15 managers is randomly selected from the three departments and rated on their technology knowledge. Then, after a 3 month training program, the same assessment rated the managers knowledge again. A "+" sign indicates an improvement, and a "-" sign indicates a decline in technology competence.

Did the in-plant training program increase the managers technical knowledge?

	Name	Before	After	Sign of Difference
	T. J. Bowers	Good	Outstanding	+
	Sue Jenkins	Fair	Excellent	+
	James Brown	Excellent	Good	−
Dropped from analysis	Tad Jackson	Poor	Good	+
	~~Andy Love~~	~~Excellent~~	~~Excellent~~	~~0~~
	Sarah Truett	Good	Outstanding	+
	Antonia Aillo	Poor	Fair	+
	Jean Unger	Excellent	Outstanding	+
	Coy Farmer	Good	Poor	−
	Troy Archer	Poor	Good	+
	V. A. Jones	Good	Outstanding	+
	Juan Guillen	Fair	Excellent	+
	Candy Fry	Good	Fair	−
	Arthur Seiple	Good	Outstanding	+
	Sandy Gumpp	Poor	Good	+

Step 1: State the null and the alternate hypothesis

H_0: $\pi \leq .50$ There has been no change in the technology knowledge base of the managers as a result of the training program

H_1: $\pi > .50$ There has been an increase in the technology knowledge base of the managers as a result of the training program

The Sign Test Example Continued

Step 2: Select the level of significance, we select .10

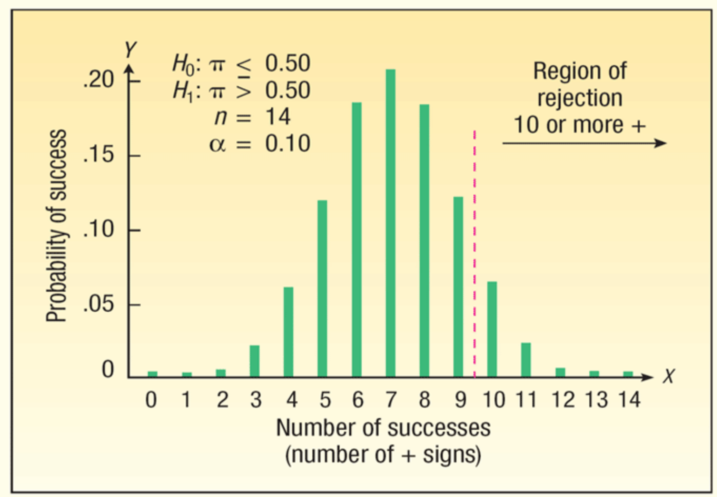
Step 3: Decide on the test statistic, it is the number of "+" signs

Step 4: Formulate a decision rule, if the number of pluses in the sample is 10 or more, the null hypothesis is rejected and the alternate hypothesis accepted

Step 5: Make a decision, reject H_0, the number of pluses is 11

Step 6: Interpret, we conclude the three-month training course was effective. It increased the technology knowledge of the managers

Number of Successes	Probability of Success		Cumulative Probability
0	0.000	↑	1.000
1	0.001		0.999
2	0.006		0.998
3	0.022		0.992
4	0.061		0.970
5	0.122		0.909
6	0.183		0.787
7	0.209	Add up	0.604
8	0.183	↑	0.395
9	0.122		0.212
10	0.061		0.090
11	0.022		0.029
12	0.006		0.007
13	0.001		0.001
14	0.000		0.000



Using the Normal Approximation to the Binomial

- For a sample of 10 or more, use the standard normal distribution and the following formula

SIGN TEST, $n > 10$ $$z = \frac{(x \pm .50) - \mu}{\sigma}$$ **[16–1]**

- If the number of pluses is more than n/2, we use formula 16-2

SIGN TEST, $n > 10$, + SIGNS MORE THAN $n/2$ $$z = \frac{(X - .50) - \mu}{\sigma} = \frac{(X - .50) - .50n}{.50\sqrt{n}}$$ **[16–2]**

- If the number of pluses is less than n/2, use formula 16-3

SIGN TEST, $n > 10$, + SIGNS LESS THAN $n/2$ $$z = \frac{(X + .50) - \mu}{\sigma} = \frac{(X + .50) - .50n}{.50\sqrt{n}}$$ **[16–3]**

Using the Normal Approximation to the Binomial Example

The market research department of Cola, Inc. has the task of taste testing a new soft drink. There are 2 versions of the drink, one sweet and one bitter. The researcher selects a random sample of 64 consumers. Each consumer will taste both the sweet cola (labeled A) and the bitter cola (labeled B) and indicate a preference. Conduct a test of hypothesis to determine if there is a difference in the preference for the sweet and bitter tastes.

Step 1: State the null and alternate hypothesis

H_0: $\pi = .50$ There is no preference

H_1: $\pi \neq .50$ There is a preference

Step 2: Select a level of significance, we use .05

Step 3: Select the test statistic, we'll use z

Step 4: Formulate the decision rule, reject H_0 if $z < -1.960$ or > 1.960

Step 5: Compute z, make decision, reject H_0, $z = 2.38$ and falls in the rejection region

$$z = \frac{(x - .50) - .50n}{.50\sqrt{n}} = \frac{(42 - .50) - .50(64)}{.50\sqrt{64}} = 2.38$$

Step 6: Interpret, we conclude consumers prefer one cola over the other

Testing a Hypothesis about a Median

- The median test is used to test a hypothesis about a population median
- Find μ and σ for a binomial distribution
- The z distribution is used as the test statistic
- The value of z is computed from formula 16-1, where x is the number of observations above or below the median
  - A value above the median is assigned a "+"
  - A value below the median is assigned a "-"
  - A value that is the same as the median is dropped from the analysis

Testing a Hypothesis about a Median Example

The U.S. Bureau of Labor Statistics reported in 2014 that the median amount spent eating out by American families is about $2,800 annually. The food editor of the *Portland Tribune* wishes to know if the citizens of Portland differ. She selected a random sample of 102 couples and found 60 spent more than $2,800 last year eating out, 40 spent less than that, and 2 spent exactly $2,800. Is it reasonable to conclude that the median amount spent this year in Portland, Oregon is not equal to $2,800?

Step 1: State the null and the alternate hypothesis

H_0: Median = $2,800

H_1: Median ≠ $2,800

Step 2: Select the level of significance, we select .10

Step 3: Select the test statistic, we'll use z

Step 4: Formulate the decision rule, reject H_0 if $z < -1.645$ or > 1.645

Step 5: Calculate z, make decision, reject the null hypothesis, $z = 1.90$ and is > 1.645

$$z = \frac{(x - .5) - .5n}{.50\sqrt{n}} = \frac{(60 - .5) - .50(100)}{.50\sqrt{100}} = 1.90$$

Step 6: Interpret, The food editor should conclude that there is a difference in the median amount spent in Portland from that reported by the BLS in 2014

The Wilcoxon Signed-Rank Test for Dependent Populations

- The Wilcoxon sign-rank test is a nonparametric test for differences between two dependent populations
- The assumption of normally distributed populations is not required
- The steps to conduct the test are
  - Rank absolute differences between the related observations
  - Apply the sign of the differences to the ranks
  - Sum negative ranks and positive ranks
  - The smaller of the two sums is the computed T value
  - Refer to Appendix B.8 for the critical value, and make a decision regarding H_0

The Wilcoxon Signed-Rank Test for Dependent Populations Example

Fricker's is a family restaurant chain located primarily in the southeastern part of the United States. It offers a full menu, but its specialty is chicken. Recently, the owner, Bernie Frick, developed a new spicy flavor for the batter in which the chicken is cooked. Before replacing the current flavor, he wants to be sure that patrons will like the spicy flavor. To begin the taste test, he selects a random sample of 15 customers. Each customer is given a piece of the current chicken and asked to rate it on a scale of 1 to 20 and then the customer is given a piece of spicy chicken and asked to rate it.

Is it reasonable to conclude that the spicy flavor is preferred?

Participant	Score Spicy Flavor	Score Current Flavor
Arquette	14	12
Jones	8	16
Fish	6	2
Wagner	18	4
Badenhop	20	12
Hall	16	16
Fowler	14	5
Virost	6	16
Garcia	19	10
Sundar	18	10
Miller	16	13
Peterson	18	2
Boggart	4	13
Hein	7	14
Whitten	16	4

The Wilcoxon Signed-Rank Test for Dependent Populations Example Continued

Step 1: State the null and the alternate hypothesis

H_0: There is no difference in the ratings of the two flavors

H_1: The spicy ratings are higher

Step 2: Select the level of significance, we select .05

Step 3: Select the test statistic, we'll use T

Step 4: Formulate the decision rule, reject the null hypothesis if the smaller of the rank sums is 25 or less

	2α						
	.15	.10	.05	.04	.03	.02	.01
	α						
n	.075	.050	.025	.020	.015	.010	.005
4	0						
5	1	0					
6	2	2	0	0			
7	4	3	2	1	0	0	
8	7	5	3	3	2	1	0
9	9	8	5	5	4	3	1
10	12	10	8	7	6	5	3
11	16	13	10	9	8	7	5
12	19	17	13	12	11	9	7
13	24	21	17	16	14	12	9
14	28	25	21	19	18	15	12
15	33	30	25	23	21	19	15
16	39	35	29	28	26	23	19
17	45	41	34	33	30	27	23
18	51	47	40	38	35	32	27

Appendix B

B.7 Wilcoxon *T* Values

	2α						
	.15	.10	.05	.04	.03	.02	.01
	α						
n	.075	.050	.025	.020	.015	.010	.005
4	0						
5	1	0					
6	2	2	0	0			
7	4	3	2	1	0	0	
8	7	5	3	3	2	1	0
9	9	8	5	5	4	3	1
10	12	10	8	7	6	5	3
11	16	13	10	9	8	7	5
12	19	17	13	12	11	9	7
13	24	21	17	16	14	12	9
14	28	25	21	19	18	15	12
15	33	30	25	23	21	19	15
16	39	35	29	28	26	23	19
17	45	41	34	33	30	27	23
18	51	47	40	38	35	32	27
19	58	53	46	43	41	37	32
20	65	60	52	50	47	43	37
21	73	67	58	56	53	49	42
22	81	75	65	63	59	55	48
23	89	83	73	70	66	62	54
24	98	91	81	78	74	69	61

The Wilcoxon Signed-Rank Test for Dependent Populations Example Concluded

Step 5: Make decision, in this case the smaller rank sum is 30, do not reject H_0
Step 6: Interpret, we cannot conclude there is a difference in the flavor ratings between the current and the spicy.

Mr. Frick should stay with the current flavor of chicken!

A	B	C	D	E	F	G	H	I
Participant	Score Spicy Flavor	Score Current Flavor	Difference in Score	Absolute Difference	Rank	Signed Rank R^+	R^-	
Arquette	14	12	2	2	1	1		
Jones	8	16	−8	8	6		6	
Fish	6	2	4	4	3	3		
Wagner	18	4	14	14	13	13		
Badenhop	20	12	8	8	6	6		
~~Hall~~	~~16~~	~~16~~	~~0~~	~~*~~	~~*~~			
Fowler	14	5	9	9	9	9		
Virost	6	16	−10	10	11		11	
Garcia	19	10	9	9	9	9		
Sundar	18	10	8	8	6	6		
Miller	16	13	3	3	2	2		
Peterson	18	2	16	16	14	14		
Boggart	4	13	−9	9	9		9	Smaller rank sum
Hein	7	14	−7	7	4		4	
Whitten	16	4	12	12	12	12		
					Sums	75	30	

The Wilcoxon Rank-Sum Test

- The Wilcoxon rank-sum test is used to test whether two independent samples came from equivalent populations
- The assumption of normally distributed populations is not required
- The population variances need not be equal either
- The data must be at least ordinal scale
- If each sample contains at least 8 observations, use the standard normal distribution as the test statistic

WILCOXON RANK-SUM TEST

$$z = \frac{W - \dfrac{n_1(n_1 + n_2 + 1)}{2}}{\sqrt{\dfrac{n_1 n_2(n_1 + n_2 + 1)}{12}}} \qquad [16\text{–}4]$$

The Wilcoxon Rank-Sum Test Example

Dan Thompson, the president of OTG airlines, recently noted an increase in the number of bags that were checked in at the gate in Atlanta. He is interested in determining whether there are more gate-checked bags from Atlanta compared with flights leaving Chicago. A sample of 9 flights from Atlanta and eight from Chicago are reported in the table.

Can we conclude that there are more checked bags for flights originating in Atlanta?

Atlanta	Chicago
11	13
15	14
10	10
18	8
11	16
20	9
24	17
22	21
25	

Ranked number of gate checked bags.

Atlanta		Chicago	
Gate-checked Bags	Rank	Gate-checked Bags	Rank
11	5.5	13	7
15	9	14	8
10	3.5	10	3.5
18	12	8	1
11	5.5	16	10
20	13	9	2
24	16	17	11
22	15	21	14
25	17		
	96.5		56.5

Atlanta Rank Sum

The Wilcoxon Rank-Sum Test Example Continued

Step 1: State the null and alternate hypothesis

H_0: The number of gate-checked bags for Atlanta is the same or less than the number of gate checked bags for Chicago

H_1: The number of gate-checked bags for Atlanta is more than the number of gate-checked bags for Chicago

Step 2: Select the level of significance, we select .05

Step 3: Select the test statistic, we use z

Step 4: Formulate the decision rule, reject H_0 if $z > 1.645$

Step 5: Compute the test statistic, make decision, do not reject H_0

$$z = \frac{W - \frac{n_1(n_1 + n_2 + 1)}{2}}{\sqrt{\frac{n_1 n_2(n_1 + n_2 + 1)}{12}}} = \frac{96.5 - \frac{9(9 + 8 + 1)}{2}}{\sqrt{\frac{9(8)(9 + 8 + 1)}{12}}} = 1.49$$

Step 6: Interpret, it appears that the number of checked bags in Atlanta are the same as those in Chicago

The Kruskal-Wallis Test: ANOVA by Ranks

- The Kruskal-Wallis one-way ANOVA by ranks is used to test whether several populations are the same
- The assumption of normally distributed populations is not required
- The samples must be independent and at least ordinal scale
- The test statistic follows the chi-square distribution, if there are at least five observations in each sample
- Compute the value of the test statistic from the following

KRUSKAL-WALLIS TEST

$$H = \frac{12}{n(n+1)}\left[\frac{(\Sigma R_1)^2}{n_1} + \frac{(\Sigma R_2)^2}{n_2} + \cdots + \frac{(\Sigma R_k)^2}{n_k}\right] - 3(n+1) \qquad \textbf{(16–5)}$$

The Kruskal-Wallis Test: ANOVA by Ranks Example

The Hospital Systems of the Carolinas operate three hospitals in the Greater Charlotte area: St. Luke's Memorial, Swedish Medical Center, and Piedmont Hospital. The director of administration is concerned about the waiting time of patients with non-life-threatening injuries that arrive during weekday evenings at the three hospitals.

Is there a difference in the waiting times at the three hospitals?

St. Luke's Memorial	Swedish Medical Center	Piedmont Hospital
56	103	42
39	87	38
48	51	89
38	95	75
73	68	35
60	42	61
62	107	
	89	

Waiting times ranked and summed

St. Luke's Memorial		Swedish Medical Center		Piedmont Hospital	
Time	Rank	Time	Rank	Time	Rank
56	9	103	20	42	5.5
39	4	87	16	38	2.5
48	7	51	8	89	17.5
38	2.5	95	19	75	15
73	14	68	13	35	1
60	10	42	5.5	61	11
62	12	107	21		
		89	17.5		
	$\Sigma R_1 = 58.5$		$\Sigma R_2 = 120$		$\Sigma R_3 = 52.5$

Waiting time rank sums

The Kruskal-Wallis Test: ANOVA by Ranks Example

Step 1: State the null and alternate hypothesis

H_0: The population distributions of waiting times are the same for the three hospitals

H_1: The population distributions of waiting times are not all the same for the three hospitals

Step 2: Select the level of significance, he selects .05

Step 3: Select the test statistic, he'll use chi-square

Step 4: Formulate the decision rule, reject H_0 if $H > 5.991$

Step 5: Calculate H, make decision, do not reject H_0

Step 6: Interpret, there is not enough evidence to conclude that there are differences in wait times

$$H = \frac{12}{n(n+1)}\left[\frac{(\Sigma R_1)^2}{n_1} + \frac{(\Sigma R_2)^2}{n_2} + \frac{(\Sigma R_3)^2}{n_3}\right] - 3(n+1)$$

$$= \frac{12}{21(21+1)}\left[\frac{58.5^2}{7} + \frac{120^2}{8} + \frac{52.5^2}{6}\right] - 3(21+1) = 5.38$$

Rank-Order Correlation

- Spearman's coefficient of rank correlation is a measure of the association between two ordinal-scale variables
- It can range from -1 up to 1
  - A value of -1 indicates perfect negative correlation
  - A value 1 indicates perfect positive correlation
  - A value of 0 indicates there is no association between the variables
- Example
- Two university staff members are asked to rank 10 faculty research proposals for funding purposes. Do the staff members rank the same proposals in the same way?

Spearman's Coefficient Continued

- The value of r_s is computed from the following formula

SPEARMAN'S COEFFICIENT OF RANK CORRELATION

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)} \quad [16\text{–}6]$$

- Provided the sample size is at least 10, we can conduct a test of hypothesis using the following formula

HYPOTHESIS TEST, RANK CORRELATION

$$t = r_s\sqrt{\frac{n - 2}{1 - r_s^2}} \quad [16\text{–}7]$$

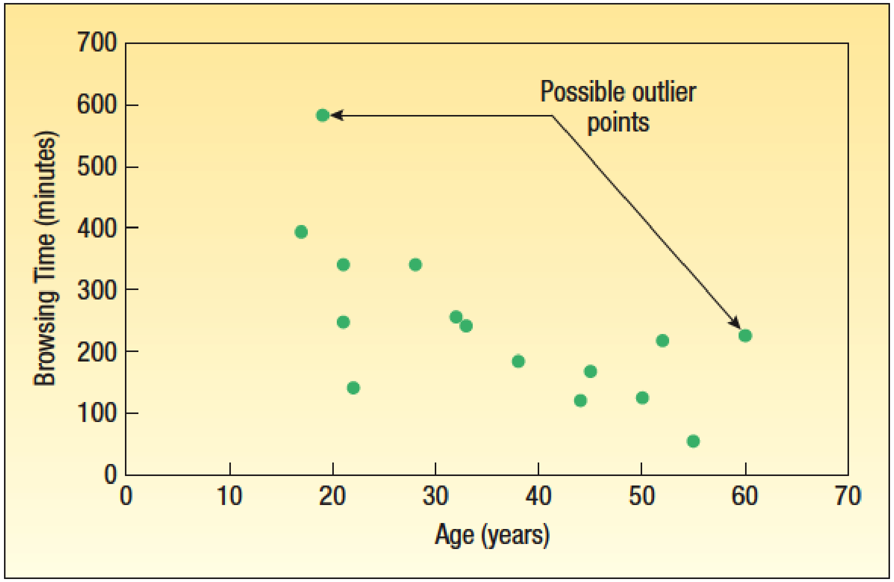
- The test statistic follows the t distribution
- There are n-2 degrees of freedom

Spearman's Coefficient Example

Recent studies focus on the relationship between the age of online shoppers and the number of minutes spent browsing on the Internet. The table below shows a sample of 15 online shoppers who actually made a purchase last week. Included is their age and time, in minutes, spent browsing on the Internet last week.

1. Draw a scatter diagram.
2. What type of association do the sample data suggest?
3. Do you see any issues with the relationship between variables?

Shopper	Age	Browsing Time (minutes)
Spina, Sal	28	342
Gordon, Ray	50	125
Schnur, Roberta	44	121
Alvear, Jose	32	257
Myers, Tom	55	56
Lyons, George	60	225
Harbin, Joe	38	185
Bobko, Jack	22	141
Koppel, Marty	21	342
Rowatti, Marty	45	169
Monahan, Joyce	52	218
Lanoue, Bernie	33	241
Roll, Judy	19	583
Goodall, Jody	17	394
Broderick, Ron	21	249



Spearman's Coefficient Example Continued

4. Find the coefficient of rank correlation.

Shopper	Age	Age Rank	Browsing Time (minutes)	Browsing Rank	d	d^2
Spina, Sal	28	6.0	342	12.5	−6.50	42.25
Gordon, Ray	50	12.0	125	3.0	9.00	81.00
Schnur, Roberta	44	10.0	121	2.0	8.00	64.00
Alvear, Jose	32	7.0	257	11.0	−4.00	16.00
Myers, Tom	55	14.0	56	1.0	13.00	169.00
Lyons, George	60	15.0	225	8.0	7.00	49.00
Harbin, Joe	38	9.0	185	6.0	3.00	9.00
Bobko, Jack	22	5.0	141	4.0	1.00	1.00
Koppel, Marty	21	3.5	342	12.5	−9.00	81.00
Rowatti, Marty	45	11.0	169	5.0	6.00	36.00
Monahan, Joyce	52	13.0	218	7.0	6.00	36.00
Lanoue, Bernie	33	8.0	241	9.0	−1.00	1.00
Roll, Judy	19	2.0	583	15.0	−13.00	169.00
Goodall, Jody	17	1.0	394	14.0	−13.00	169.00
Broderick, Ron	21	3.5	249	10.0	−6.50	42.25
					Sum d^2	965.50

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)} = 1 - \frac{6(965.5)}{15(15^2 - 1)} = 1 - 1.724 = -0.724$$

Spearman's Coefficient Example Concluded

5. Conduct a test of hypothesis to determine if there is a negative association between the ranks.

Step 1: State the null and alternate hypothesis

H_0: The rank correlation in the population is zero

H_1: There is a negative association among the ranks

Step 2: Select the level of significance, we select .05

Step 3: Select the test statistic, we use t

Step 4: Formulate the decision rule, reject H_0 if $t < -1.771$

Step 5: Make decision, reject the null hypothesis, $t = -3.784$

$$t = r_s\sqrt{\frac{n-2}{1-r_s^2}} = -0.724\sqrt{\frac{15-2}{1-(-0.724)^2}} = -3.784$$

Step 6: Interpret, there is evidence of a negative association between the age of the Internet shopper and the time spent browsing the Internet